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AN INITIAL-VALUE METHOD FOR  
THE SIMPLEST PROBLEM IN  
THE CALCULUS OF VARIATIONS

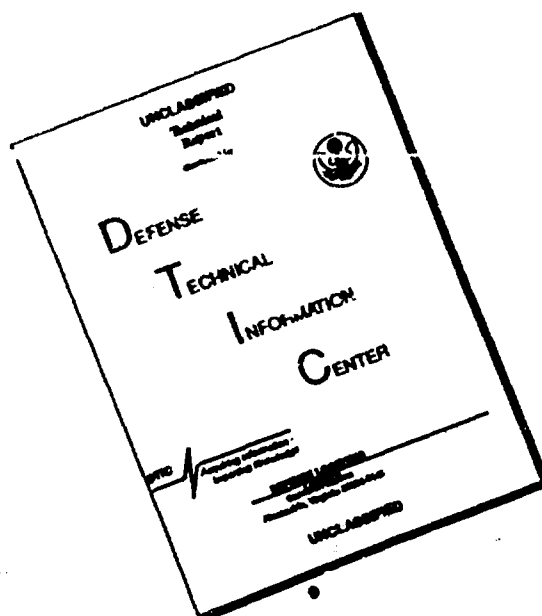
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PREFACE AND SUMMARY

A method for converting certain optimization problems in the calculus of variations into Cauchy problems is presented. No direct use of Euler equations or dynamic programming is involved. This provides a new conceptual framework for variational problems with both computational and analytical advantages.

This Memorandum may be of interest to applied mathematicians, control theorists, and numerical analysts.

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## I. INTRODUCTION

In recent years methods have been developed for transforming various classes of functional equations into initial-value problems. One of the motivations is that current computing machines can frequently handle such Cauchy problems with accuracy and dispatch.<sup>(1,2)</sup> Among the classes considered have been two-point boundary-value problems,<sup>(3)</sup> Fredholm integral equations,<sup>(4)</sup> and multi-point boundary-value problems.<sup>(5)</sup>

It will be shown that certain variational problems can be transformed directly into Cauchy problems, without using Euler equations or dynamic programming. This, in effect, provides a new conceptual framework for attacking variational problems with obvious computational advantages and attractive analytical features for future exploitation.

## II. DERIVATION OF THE EQUATIONS

Consider the minimization of the integral

$$I(w) = \int_a^T f(y, w, \dot{w}) dy, \quad (1)$$

where

$$w = w(y), \quad a \leq y \leq T,$$

and

$$\dot{w} = \frac{dw}{dy}. \quad (2)$$

Furthermore, let

$$w(a) = c, \quad (3)$$

with  $w(T)$  being free. Let

$$w = x = x(y, a, c) \quad (4)$$

be the minimizer. For an arbitrary admissible variation  $\eta(y)$ ,  
 $a \leq y \leq T$ , with

$$\eta(a) = 0, \quad (5)$$

a necessary condition on  $x$  is that the first variation of the integral in Eq. (1) be zero. This leads to the equation

$$\int_a^T [\beta(y, x, \dot{x}) \eta + \alpha(y, x, \dot{x}) \dot{\eta}] dy = 0, \quad (6)$$

where

$$\alpha(y, w, \dot{w}) = \frac{\partial f}{\partial \dot{w}} , \quad (7)$$

and

$$\beta(y, w, \dot{w}) = \frac{\partial f}{\partial w} . \quad (8)$$

Suitable integration by parts in Eq. (6) and use of the fundamental lemma of the calculus of variations would lead to the Euler equation. The procedure to be used below is different.

The choice of the admissible variation

$$\dot{\eta}(y) = \begin{cases} 1, & a \leq y \leq t, \\ 0, & t < y \leq T, \end{cases} \quad (9)$$

transforms Eq. (6) into the differential integral equation

$$\int_a^t \alpha(y, x, \dot{x}) dy + \int_a^T \beta(y, x, \dot{x}) \eta dy = 0 . \quad (10)$$

From Eqs. (5) and (9), assuming continuity, it follows that

$$\eta(y) = k(t, y, a) = \min(t-a, y-a) . \quad (11)$$

From Eqs. (10) and (11) it is seen that

$$\int_a^t \alpha(y, x, \dot{x}) dy + \int_a^T k(t, y, a) \beta(y, x, \dot{x}) dy = 0 . \quad (12)$$

Differentiating both sides of Eq. (12) with respect to the variable  $t$  leads to the relation

$$\alpha(t, x(t), \dot{x}(t)) + \int_a^T k_t(t, y, a) \beta(y, x, \dot{x}) dy = 0. \quad (13)$$

From Eq. (11) we find

$$k_t(t, y, a) = \begin{cases} 0, & y \leq t, \\ 1, & y > t. \end{cases} \quad (14)$$

From Eqs. (13) and (14) it follows that

$$\alpha(t, x, \dot{x}) + \int_t^T \beta(y, x, \dot{x}) dy = 0. \quad (15)$$

We wish to study the minimizer  $x$  as a function of  $c$  and  $a$ . Thus, when necessary, we shall write

$$x = x(y, a, c), \quad a \leq y \leq T, \quad -\infty < c < \infty, \quad (16)$$

and Eq. (15), in full, will be written as

$$\alpha(t, x(t, a, c), \dot{x}(t, a, c)) + \int_t^T \beta(y, x(y, a, c), \dot{x}(y, a, c)) dy = 0. \quad (17)$$

Another differentiation with respect to  $t$  of Eq. (17) would yield the Euler equation. We shall proceed along different lines. In Eq. (17), as well as in further developments, the dot refers to partial differentiation with respect to the first variable. The condition specified

by Eq. (4) is equivalent to

$$x(a, a, c) = c, \quad (18)$$

or, written differently,

$$x(t, t, c) = c. \quad (18)$$

Differentiation of Eq. (17) with respect to  $a$  leads to the linear integro-differential equation for the partial derivative  $x_a$ ,

$$\begin{aligned} \alpha_w(t, x, \dot{x}) x_a + \alpha_w(t, x, \dot{x}) \dot{x}_a + \int_t^T \{ \beta_w(y, x, \dot{x}) x_a \\ + \beta_w(y, x, \dot{x}) \dot{x}_a \} dy = 0, \end{aligned} \quad (19)$$

where

$$x_a = \frac{\partial x}{\partial a}(t, a, c). \quad (20)$$

Taking the total derivative with respect to  $a$  of both sides of Eq. (18) results in the formula

$$\dot{x}(a, a, c) + x_a(a, a, c) = 0. \quad (21)$$

Differentiation of Eq. (17) with respect to  $c$  leads to the linear integro-differential equation for the partial derivative  $x_c$ ,

$$\begin{aligned} \alpha_w(t, x, \dot{x}) x_c + \alpha_w(t, x, \dot{x}) \dot{x}_c + \int_t^T \{ \beta_w(y, x, \dot{x}) x_c \\ + \beta_w(y, x, \dot{x}) \dot{x}_c \} dy = 0. \end{aligned} \quad (22)$$

In Eqs. (19) and (22)

$$\alpha_w(y, w, \dot{w}) = \frac{\partial \alpha}{\partial w}(y, w, \dot{w}) , \quad (23)$$

where  $\alpha$  is defined in Eq. (7). The variable  $\alpha_w$  is similarly defined.

Taking the total derivative with respect to  $c$  of both sides of Eq. (18) results in

$$x_c(a, a, c) = 1 . \quad (24)$$

It is seen from Eqs. (19) and (22) that the functions  $x_a$  and  $x_c$  satisfy identical linear homogeneous integro-differential equations. The initial conditions given by Eqs. (21) and (24), though, are different for these functions. Assuming uniqueness, it follows that

$$x_a(t, a, c) = - \dot{x}(a, a, c) x_c(t, a, c) . \quad (25)$$

Define

$$p(a, c) = \dot{x}(a, a, c) . \quad (26)$$

Equation (25) becomes

$$x_a(t, a, c) = - p(a, c) x_c(t, a, c) , \quad a \leq t . \quad (27)$$

Equation (27) is the partial differential equation satisfied by  $x$ . It has to be solved with the initial condition specified by the second Eq. (18). We now have to determine  $p(a, c)$  before Eqs. (27) and (18) can be solved.

Define

$$\alpha(a, x(a, a, c), \dot{x}(a, a, c)) = r(a, c) . \quad (28)$$

From Eqs. (18), (26), and (28), it follows that

$$r(a,c) = \alpha(a,c, p(a,c)) . \quad (29)$$

Assuming that Eq. (29) is invertible,  $r(a,c)$  determines  $p(a,c)$ . We will now derive the partial differential equation and associated initial condition satisfied by  $r(a,c)$ . When  $t = a$ , Eqs. (17) and (29) lead to

$$r(a,c) + \int_a^T \beta(y, x(y,a,c), \dot{x}(y,a,c)) dy = 0 . \quad (30)$$

Taking partial derivatives with respect to  $a$  on both sides of Eq. (30) leads to the equation

$$\begin{aligned} r_a(a,c) - \beta(a, x(a,a,c), \dot{x}(a,a,c)) + \int_a^T \{ \beta_w(y,x,\dot{x}) x_a \\ + \beta_{\dot{w}}(y,x,\dot{x}) \dot{x}_a \} dy = 0 . \end{aligned} \quad (31)$$

From Eqs. (18), (26), (27), and (31), it follows that

$$\begin{aligned} r_a(a,c) - \beta(a,c, p(a,c)) - p(a,c) \int_a^T \{ \beta_w(y,x,\dot{x}) x_c \\ + \beta_{\dot{w}}(y,x,\dot{x}) \dot{x}_c \} dy = 0 . \end{aligned} \quad (32)$$

Taking partial derivatives with respect to  $c$  on both sides of Eq. (30) leads to the equation

$$r_c(a,c) + \int_a^T \{ \beta_w(y,x,\dot{x}) x_c + \beta_{\dot{w}}(y,x,\dot{x}) \dot{x}_c \} dy = 0 . \quad (33)$$

Equations (32) and (33) result in the desired formula

$$r_a(a,c) + p(a,c)r_c(a,c) = \beta(a,c,p(a,c)) . \quad (34)$$

From Eq. (30) it is evident that

$$r(T,c) = 0 . \quad (35)$$

The partial differential Eq. (34), the finite Eq. (24), and Eq. (35) constitute an initial-value problem for the functions  $r(a,c)$  and  $p(a,c)$ .

In practice the function  $r$  could be determined first and stored, and then the function  $x$  could be determined. To avoid storage problems in the digital computer, the functions  $r$  and  $x$  could be determined simultaneously.

### III. DISCUSSION

A method for directly obtaining the initial-value problem associated with the simplest problem in the calculus of variations has been presented. It is contained in Eqs. (27), (18), (34), (35), and (29). Extensions of this method to more general problems in the calculus of variations, as well as in optimal control theory, are under study. (6)

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